# The non-adiabatic classical geometric phase and its bundle-theoretic interpretation

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# I. NOTATION

| | |
|---|---|
| $\sigma_{Borel}$ | Borel $\sigma$-algebra of a topological space |
| $U_T$ | Koopman unitary of the dynamical system $(X\ ,\ \sigma\ ,\ \mu\ ,\ T)$ |
| P(M,G) | principal bundle with total-space P, base-space M and structure-group G |
| $\mathcal{S}(\mathcal{H})$ | space of the normalized vectors in $\mathcal{H}$ |
| $P\mathcal{H}$ | space of the rays in $\mathcal{H}$ |
| $C_p(M)$ | space of the p-based loops in M |
| $\omega_{Stiefel}$ | Stiefel connection on a universal bundle |
| $\tau_{\gamma}^{\omega}$ | holonomy of the loop $\gamma$ w.r.t. the connection $\omega$ |

## II. INTRODUCTION

While the investigation of Berry phase has led to [1], [2], [3] :

1. a geometric interpretation of the phenomenon in terms of holonomies of a suitable connection over a suitable principal bundle over the parameters'-space (performed by Barry Simon)

2. a nonadiabatic generalization of its (performed by Yakir Aharonov and Jeeva Anandan)

3. a geometric interpretation of the nonadiabatic Aharonov-Anandan phase in terms of holonomies of the Stiefel connection on the U(1)-universal-bundle (performed by Yakir Aharonov and Jeeva Anandan themselves and refined and improved by Arno Bohm and Alí Mostafazadeh)

only the first of these conceptual step has been performed as to classical Hannay phase [4] whose bundle-theoretic interpretation has been shown by Ennio Gozzi and William D. Thacker [5].

Since Hannay's analysis applies to a very particular class of dynamical systems (the integrable hamiltonian ones) I will follow here a radically different approach consisting in:

1. considering the holonomies of the Stiefel connection on the U(1)-universal-bundle associated to the Koopman representation of an arbitrary classical dynamical system

2. investigating their physical meanings as non adiabatic classical geometric phases

3. obtaining the Hannay's result as a particular case

## III. HOLONOMIES IN THE KOOPMAN REPRESENTATION

Given an arbitrary classical dynamical system $(X\,,\,\sigma\,,\,\mu\,,\,T)$ (where hence $(X\,,\,\sigma\,,\,\mu)$ is a classical probability space and T is an automorphism of its) [6], [7] let us introduce the **Koopman unitary** $U_T : \mathcal{H} \mapsto \mathcal{H}$:

$$(U_T f)(x) \;:=\; (f \circ T)(x) \;\; f \in \mathcal{H} \tag{3.1}$$

with:

$$\mathcal{H} \;:=\; L^2(X, \mu) \tag{3.2}$$

Let us now consider the **U(1)-universal-bundle** (cfr. e.g. the mathematical appendix [8] of [2] or [9]) $\mathcal{S}(\mathcal{H})(P(\mathcal{H})\,,\,U(1))$ whose **base-space** (i.e. the **U(1)-classifying-space**) is the space of all the rays in $\mathcal{H}$, whose **total space** $\mathcal{S}(\mathcal{H})$ is the unit sphere in $\mathcal{H}$:

$$\mathcal{S}(\mathcal{H}) \;:=\; \{|\psi>\in \mathcal{H} \;\; <\psi|\psi> = \; 1\} \tag{3.3}$$

and whose **structure group** is clearly U(1).

Endowed the U(1)-universal bundle $\mathcal{S}(\mathcal{H})(P\mathcal{H}\,,\,U(1))$ with the Stiefel connection $\omega_{Stiefel}$ let us denote by $HOL_{Stiefel}(|\psi>)$ the **holonomy group in** $|\psi>$ of $\omega_{Stiefel}$:

$$HOL_{Stiefel}(|\psi>) \;:=\; \{e^{i\theta} \in U(1) : \tau_{\gamma}^{\omega_{Stiefel}}(|\psi>) = e^{i\theta}|\psi>\} \;\; \gamma \in C_{|\psi><\psi|}(P\mathcal{H}) \tag{3.4}$$

where $C_{|\psi><\psi|}(P\mathcal{H})$ is the space of the loops in $P\mathcal{H}$ at $|\psi><\psi|$ while $\tau_{\gamma}^{\omega_{Stiefel}} : \pi^{-1}(|\psi><\psi|) \mapsto \pi^{-1}(|\psi><\psi|)$ is the transformation of the fibre in $|\psi><\psi|$ associated to a loop $\gamma : [0,1] \mapsto P\mathcal{H}\, \gamma(0) = \gamma(1) = |\psi><\psi|$ in $P\mathcal{H}$

**Example III.1**

UNCOUPLED OSCILLATORS

Let us consider the hamiltonian dynamical system with configuration space $\mathbb{R}^2$ and hamiltonian $H : T^{\star}\mathbb{R}^2 \mapsto \mathbb{R}$ given by:

$$H(\vec{q}, \vec{p}) \;:= H_1 + H_2 \tag{3.5}$$

$$H_i(\vec{q_i}, \vec{p_i}) \;=\; \frac{1}{2}(\vec{p_i}^2 + \omega_i^2 \vec{q_i}^2) \;\; i = 1, 2 \tag{3.6}$$

The system is obviously integrable its **action variables**:

$$\vec{I} \;:=\; (\frac{h_1}{\omega_1}\,,\,\frac{h_2}{\omega_2}) \tag{3.7}$$

being constants of the motion:

$$\frac{d\vec{I}}{dt} \;=\; 0 \tag{3.8}$$

while the **angle variables** $\vec{\phi}$ evolve on the 2-torus $T^2$ according to:

$$T_t(\vec{\phi}) \;=\; \vec{\phi} + \vec{\omega} t \tag{3.9}$$

where obviously $\vec{\omega} := (\omega_1\,,\,\omega_2)$.

Since we have seen that our dynamical system may be seen as $(T^2, \sigma_{Borel}, d\mu(\phi_1, \phi_2) := \frac{d\phi_1 d\phi_2}{(2\pi)^2}\,,\, T_t)$ the passage to the Koopman representation involves the unitary operator $U_{T_t}$ over $\mathcal{H} := L^2(T^2, d\mu(\phi_1, \phi_2))$ identified by its action on the basis:

$$\mathbb{E} \;:=\; \{\, |\vec{n}> := \exp(i\vec{n}\cdot\vec{\phi}) \;\; \vec{n} \in \mathbb{Z}^2 \,\} \tag{3.10}$$

given by:

$$U_{T_t}|\vec{n}> = \; \exp(i\vec{n}\cdot\vec{\omega}t)\,|\vec{n}> \tag{3.11}$$

The geometrical phases in analysis emerges as the **holonomy group** $HOL_{Stiefel}(|n>)$ **in** $|\vec{n}>$ of $\omega_{Stiefel}$ where the restriction to the vectors of the basis will be clarified in the next section

**Example III.2**

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Let us consider the automorphism of the torus $T^2 \;:=\; \mathbb{R}^2/\mathbb{Z}^2$ identified by the linear operator on $\mathbb{R}^2$ whose matrix w.r.t. the canonical basis is given by:

$$C \;:=\; \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \tag{3.12}$$

The passage to the Koopman representation involves again the unitary operator $U_C$ over $\mathcal{H} := L^2(T^2, d\mu(\phi_1, \phi_2))$ identified by its action on the basis:

$$\mathbb{E} \;:=\; \{\, |\vec{n}> := \exp(i\vec{n}\cdot\vec{\phi}) \;\; \vec{n} \in \mathbb{Z}^2 \,\} \tag{3.13}$$

given by:

$$U_C|\vec{n}> = \ |C\,\vec{n}> \tag{3.14}$$

As in the example III.2, the geometrical phases in analysis emerges as the **holonomy group** $HOL_{Stiefel}(|n>)$ **in** $|\vec{n}>$ of $\omega_{Stiefel}$ where the restriction to the vectors of the basis will be clarified in the next section.

## IV. THE PHYSICAL MEANING OF THE CLASSICAL GEOMETRIC PHASE IN TERM OF MOVING FRAMES

In the previous section we have introduced, in the Koopman representation of a classical dynamical system, the mathematical setting underlying, in a quantum context, the Aharonov-Anandan (non-adiabatic) quantum geometric phase.

In this paragraph we will explore the physical meaning of the resulting (non adiabatic) classical geometric phase.

Let us start observing that, contrary to the quantum case, $P\mathcal{H}$ doesn't represent the pure states of the system but simply a particular class of physical observables observed up to a phase [1] .

It is important, with this regard, to observe, that, contrary to the quantum case, the neglected phase *do have here physical meaning*, since $|\psi >$ and $\exp(i\theta)|\psi >$ are physically distinguishable.

Let us observe, furthermore, that the similarity between the evolution equation of an **observable** $|\psi > \in \mathcal{H}$:

$$|\psi(t=1)> = \; U_T|\psi(t=0)> \tag{4.1}$$

and the evolution-equation of Quantum Mechanics in the **Schrödinger picture** is deceptive since in our classical Koopmanian situation we are in the **Heisenberg picture** according to which the observables evolve with time while the **state** $\omega_\mu : L^2(X \, , \, d\mu) \mapsto \mathbb{C}$:

$$\omega_\mu(f) \; := \; \int_X f \, d\mu \tag{4.2}$$

doesn't evolve with time.

Considered a basis of $\mathcal{H}$:

$$\mathbb{E} \; := \; \{ \, |n> \, , \, n \in \mathbb{N} \} \tag{4.3}$$

let us consider the associated family of projectors

$$P\mathbb{E} \; := \; \{ \, |n><n| \, , \, n \in \mathbb{N} \} \tag{4.4}$$

It determines the **frame** we use to decompose and consequentially analyze the particular class of physical observables constitued by the elements of $\mathcal{H}$.

[1] We are admitting here a class of physical observables $\mathcal{H}$ broader than the usually accepted one $L^\infty(X, \mu)$

When such a **frame** is allowed **to move** for a second following a loop $\gamma \in C_{|n><n|}(P\mathcal{H})$:

$$\gamma : [0,1] \mapsto P\mathcal{H} \; : \; \gamma(t=0) \; = \; \gamma(t=1) \; = \; |n><n| \tag{4.5}$$

the way the physical observables of $\mathcal{H}$ observed, through it, appear to us in the time interval $(0,1)$ is the net result of two effects:

1. the real dynamical evolution of the frame

2. the motion of the frame we are performing making it to follow the prescribed loop

At time t=1 the state of affairs of our observables aren't identical to that we would have if we hadn't moved our frame: they differ precisely by the geometric phase $\tau_{\gamma}^{\omega_{Stiefel}}(|n>) := \exp(i\theta)$ as it may be more clearly visualized in the examples:

**Example IV.1**

NON ADIABATIC GEOMETRIC PHASE OF THE UNCOUPLED OSCILLATORS

At time $t=1$ we will have that:

$$|\vec{n}(t=1)> = \; e^{i\theta} U_{T_t} |\vec{n}> = \; \exp(i\vec{n} \cdot \vec{\omega} t \, + \, \theta) \, |\vec{n}> \tag{4.6}$$

**Example IV.2**

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At time $t=1$ we will have that:

$$|\vec{n}(t=1)> = \; \exp(i\theta) U_C |\vec{n}> = \; \exp(i\theta) |C \, \vec{n}> \tag{4.7}$$

## V. THE RECOVERING OF HANNAY ANGLE IN THE ADIABATIC CASE

The link between the fibre-bundle setting underlying the Aharonov-Anandan (non-adiabatic) quantum geometric phase and the fibre-bundle setting underlying Berry adiabatic quantum geometric phase (discovered by Barry Simon) has been mathematically characterized by A. Bohm amd Alí Mostafazadeh simply through a suitable pull-back of the involved fibre bundle.

It is then reasonable that both the fibre-bundle setting underlying Hannay adiabatic classical geometric phase and its link with the non adiabatic classical geometric phase discusssed in the previous sections may be simply obtained mimicking the Bohm-Mostafazadeh-pullback.

Letc us consider an Hamiltonian integrable dynamical system [10] with motion-equation, in the angle-action variables $(\vec{I}, \vec{\phi})$, given by:

$$\frac{d\vec{I}}{dt} = 0 \tag{5.1}$$

$$\frac{d\vec{\phi}}{dt} = \vec{\omega}(I) \tag{5.2}$$

with:

$$\vec{\omega}(I) = \frac{\partial H}{\partial \vec{I}} \tag{5.3}$$

If the dynamical system in analysis has n degrees of freedoms, and hence phase space 2-n dimensional, its Koopman description involves the introduction of the Hilbert space:

$$\mathcal{H} := L^2(T^n, \frac{d\vec{\phi}}{(2\pi)^n}) \tag{5.4}$$

and of the Koopman unitary $U_t$ identified by its action on the basis:

$$\mathbb{E} := \{ |\vec{n} >:= \exp(i\vec{n} \cdot \vec{\phi}) \ \ \vec{n} \in \mathbb{Z}^n \} \tag{5.5}$$

given by:

$$U_t|\vec{n} > = \exp(i\vec{n} \cdot \vec{\omega} t) \, |\vec{n} > \tag{5.6}$$

Let us suppose to alter the hamiltonian $H \to H(\vec{R})$ making the parameter $\vec{R}$ to evolve adiabatically realizing a loop in a suitable parameter space M.

In the adiabatic limit the basis:

$$\mathbb{E}_{\vec{R}} \ := \ \{ \, |\vec{n}, \vec{R} > \quad \vec{n} \in \mathbb{Z}^n \, , \vec{R} \in M \} \tag{5.7}$$

continues to be formed by eigenvectors of $U_t$.

Let us then introduce the following map $f_{\vec{n}} : \mathbb{Z}^n \times M \ \mapsto \ P\mathcal{H}$:

$$f_{\vec{n}}(\vec{R}) \ := \ |\vec{n}, \vec{R} >< \vec{n}, \vec{R}| \tag{5.8}$$

Obviously $f_{\vec{n}}$ may also be seen as a $\vec{n}$-parametrized family of maps with domain M and codomain $P\mathcal{H}$.

Let us then introduce the pullback-bundle $f_{\vec{n}}^{\star}\mathcal{S}(\mathcal{H})$ of the Stiefel bundle $\mathcal{S}(\mathcal{H})(P\mathcal{H}, U(1))$ by $f_{\vec{n}}$ and let us denote by $f^{\star}\omega_{Stiefel}$ the connection on the principal bundle $f_{\vec{n}}^{\star}\mathcal{S}(\mathcal{H})$ induced by the Stiefel connection $\omega_{Stiefel}$ through the pull-back operation.

Given a loop $\gamma \in C_{\vec{R}=0}(M)$ the associated Hannay phase is then $\tau_{\gamma}^{f^{\star}\omega_{Stiefel}}$.

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